Scaling limits of random planar maps based on minicourse by N. Holden

Notes taken by Pantelis Tassopoulos

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These notes are produced entirely from the minicourse I followed at the Probability at Warwick conference in July 2025, `https://warwick.ac.uk/fac/sci/statistics/news/patw_summer_school/`, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. All errors are almost surely mine. Please send any corrections to pkt28@cam.ac.uk.

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1 Introduction

Why study random planar maps?

It turns out they have deep connections to theoretical physics (statistical mechanics, string theory, conformal field theory) and are used to construct canonical models of random surfaces in two dimensions. Their scaling limits also possess a rich structure. For instance, scaling limits of random planar maps are critical in the construction of the **Brownian map**, a universal limiting object, can be considered as a generalisation of Brownian motion to two spatial dimensions, see [LG13]. Moreover, convergence has been established to other canonical models for random surfaces in two dimensions, namely **Liouville quantum gravity (LQG)** surfaces.

In these notes, a brief introduction planar maps will be given as well as a discussion on some topologies under which we will consider convergence. Next, we will introduce canonical continuum objects such as the Gaussian free field and Liouville quantum gravity surfaces that appear in the scaling limits of random planar maps in the various topologies discussed above. Finally, sketches of the arguments in the above convergence results will be given.

The reader is encouraged to continue their journey into this vast and beautiful subject by moving on the the numerous references that have been mentioned herein.

2 Random planar maps (RPMs)

What is a random planar map, anyway?

First we discuss what a (deterministic) planar map is. Informally, a planar map is a finite graph, which is connected (that is every two vertices on the planar map are connected by a path), and is viewed as 'embedded' in the plane (examples include triangulations, quadrangulations, etc.). We also consider two such 'embedded graphs' as equivalent if one can continuously deform the edges and vertex locations from one to the other without that resulting in crossing of edge (see Figure 1 for an example of two non-exivalent planar maps $\mathcal{M}, \mathcal{M}'$). More precisely, we make the following definition.

Definition 2.1 (Planar map)**.** *A planar map is a proper embedding of a finite connected (multi-graph) in the sphere* $S^{2\,a}$*, considered up to orientation-preserving homeomorphisms. For a planar map* $\mathcal{M}$*, we call its vertices* $V(\mathcal{M})$*, underlineedges* $E(\mathcal{M})$*, faces* $F(\mathcal{M})$ *and a distinguished edge,* $\mathbb{e}$*, its oriented root edge with the face to its right, the corresponding root face.*

[a] we identify S^2 with $\mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex plane.

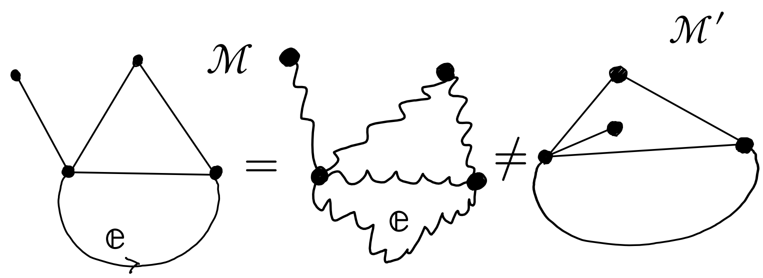


Figure 1: An illustration of distinct planar maps $\mathcal{M}$ and $\mathcal{M}'$ with distinguished root edge $\mathbb{e}$.

Now, we will discuss how one might add 'randomness' to the above construction. This is most easily done when we can restrict ourselves to classes of planar maps that are finite. This allows us to sample planar maps 'uniformly'. In principle one can sample from any arbitrary finite sub-collection of planar maps by considering the uniform, though this is hardly motivated and will unlikely lead to interesting results when one considers scaling limits. However, one does obtain interesting results by imposing certain restrictions on the graph structure of the planar maps, which we will discuss below.

Examples 2.2 (Uniform RPMs)**.** *For* $n \geq 1$*, let*

$$\mathcal{S}_n = \{m : m \text{ is a planar map with } n \text{ edges}\} .$$

Since this set is finite, one can consider the random planar map M *sampled according to the uniform measure, by setting for any* $m \in \mathcal{S}_n$*,*

$$\mathbb{P}(M = m) = \frac{1}{|\mathcal{S}_n|} ,$$

where $|\cdot|$ *denotes the cardinality of a (finite) set.*

The motivation for this notion of uniform random planar map comes from asking the same question in one spatial dimension, namely, how does one sample a path uniformly at random? Again, to restrict ourselves to finite collection of paths, one might consider 'walks' starting from the origin making bounded jumps at every time step until a finite time horizon. For simplicity, we can take the jumps at every time interval to be unit upward, or downward jumps. Then, in this case the uniform measure on such paths turns out to be the simple random walk on $\mathbb{Z}$.

The simple random walk and its variants have been well studied and there is a vast literature on convergence results, in particular scaling limits to continuum objects. A classical example is the convergence on paths of the simple random walk under Brownian scaling to a Brownian motion, also known as Donsker's theorem (see Figure2).

Theorem 2.3 (Donsker's invariance principle)**.** *Let $X_1, X_2, \cdots$ be iid $\mathbb{R}-$valued integrable random variables with law μ, such that $\mathbb{E}[X_1] = 0$, and variance $\sigma^2 \in (0, \infty)$. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $S_t = (1 - \{t\})S_{[t]} + \{t\}S_{[t]+1}$, where $\{t\} = t - [t]$ and $[t]$ is the integer part of $t \geq 0$. Now, define*

$$S_t^{[N]} = \frac{S_{tN}}{\sqrt{\sigma^2 N}}$$

for $0 \leq t \leq 1$. Then, $\left(S_t^{[N]}, 0 \leq t \leq 1\right)$ converges weakly to $(B_t, 0 \leq 1 \leq 1)$, that is to a standard Brownian motion. More explicitly, we have for all continuous (in the local uniform topology) and bounded functionals $F : C([0,1], \mathbb{R}) \to \mathbb{R}$

$$\mathbb{E}\left[F(S^{[N]})\right] \overset{N \to \infty}{\longrightarrow} \mathbb{E}\left[F(B)\right].$$

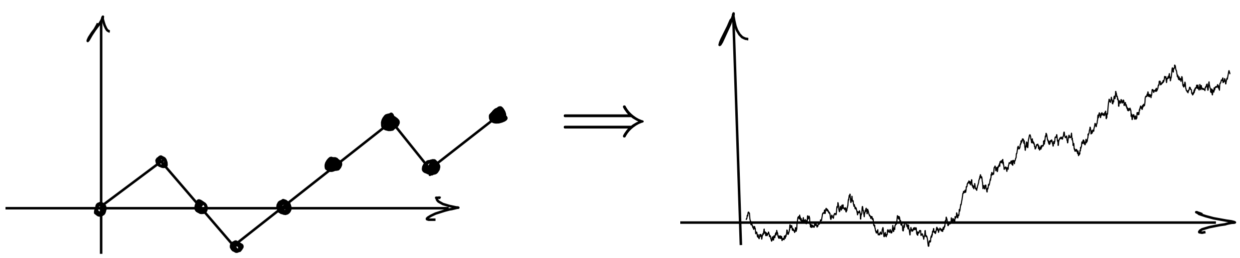


Figure 2: **Left**: a simple random walk on a finite interval. **Right**: a Brownian motion on another bounded interval. This figure is meant to illustrate Donsker's theorem: the convergence of the (linearly interpolated and rescaled) simple random walks to a Brownian motion. The convergence is established as one 'zooms out' in the left picture and appropriately rescales, and the resulting in the 'packing more and more' randomness on a given interval which establishes convergence with the aid of the strong law of large numbers.

Now, one might naturally ask, is there an analogue for surfaces? What is a 'uniform' surface?

One way to approach these questions is to first consider a discretisation of space (by some sort of lattice or polygonal tiling), and then consider uniform random planar maps. However, there is another notion of 'uniform' planar map, one which takes into account the number spanning trees of a planar map, which we turn to.

Definition 2.4 (Spanning tree)**.** *A spanning tree of a planar map m is a set of edges which is spanning, connected and has no cycles.*

Equipped with this notion of spanning tree (see Figure 3 for an illustration of a spanning tree of a planar map $\mathcal{M}$), we now consider the so-called 'tree-weighted random planar maps'.

Examples 2.5 (Tree-weighted RPMs). *For $n \geq 1$, a planar map $m \in \mathcal{S}_n$, let*

$$T(m) = |\{ \text{ spanning trees admitted by } m\}| .$$

Then, one can consider the random planar map M sampled by setting for any $m \in \mathcal{S}_n$,

$$\mathbb{P}(M = m) \sim T(m) .$$

This measure can be realised as the first marginal of the uniform atomic measure on pairs $(\mathcal{S}_n, T)$ of planar maps with n edges and spanning trees thereof, respectively.

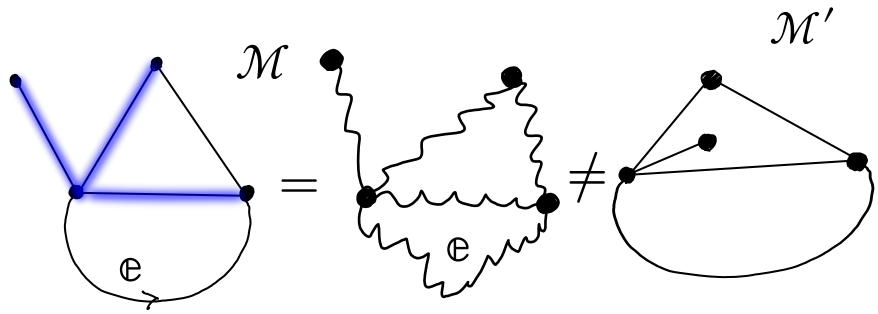


Figure 3: An illustration of distinct planar maps $\mathcal{M}$ and $\mathcal{M}'$ with a spanning tree (blue) of $\mathcal{M}$ indicated.

Tree-weighted planar maps arise naturally in multiple contexts.

- In the context of graph theory, one can write for a planar map m,

$$T(m) = \det\Delta(m) ,$$

  which is called the 'Laplacian determinant', that is the determinant of the discrete Laplacian matrix obtained from the adjacency matrix of m (or the degree minus the adjacency matrix);

- $T(m)$ is also related to other statistical physics models and scaling limits;

- perhaps most importantly for us, it features in the Mullin bijection, [Mul67], which states that there is a bijection between tree-distinguished planar maps and discrete 'loops'

$$(M, T) \overset{\text{bijection}}{\longleftrightarrow} \text{ walk } (W_j)_{j \in \{1, \cdots, 2n\}} \text{ on } \mathbb{Z}_+^2 , W_0 = W_{2n} = 0, \|W_{j+1} - W_j\|_1 = 1 . \tag{2.1}$$

Now, armed with some notions of uniform random planar maps, the next natural question is in what sense can we speak of random planar maps converging in the scaling limit? This requires us to specify the underlying topology of convergence, which we discuss next.

3 Convergence of random variables

We now discuss various topologies under which we will consider the convergence of random planar maps. The point for each topology is to capture some underlying geometric or 'physical' characteristic of our discrete families of planar maps and establish convergence. The setup of interest will be the following, which is in line with the standard framework of weak convergence on metric spaces one encounters in Probability.

- A metric space (S, d).

- Random variables $X_1, X_2, \cdots$ with values in S.
- We say that a family $(X_n)_{n \in \mathbb{N}}$ of S-valued random variables converges in law to X if for all continuous bounded $f : S \to [0, \infty)$,

$$\mathbb{E}[f(X_n)] \overset{n \to \infty}{\longrightarrow} \mathbb{E}[f(X)] \, .$$

Now, what will be the most natural choice of metric space (S, d)? There is no unique choice, and we see that different choices lead to different topologies of convergence. In the context, of random planar maps, we will be considering three topologies:

- Metric topology (graph distance)
- Peanosphere topology (statistic of statistical physics models)
- Conformal topology

The first is the most abstract in some sense, as it involves considering our planar maps as (compact) metric spaces and establishing converece in an appropriately defined topology. The latter is inspired from physics where one considers a physically relevant 'observable', or statistic of our random planar maps and considers convergence, and the final topology will rely on some notion of discrete conformal embeddings of our random planar maps (which can be defined in multiple ways). We start with the metric topology.

3.1 The metric topology

We snow briefly discuss the metric topology. Consider the set

$$\mathcal{S} = \{X : X \text{ compact metric space }\} \, .$$

A key observation is that the above collection of all compact metric spaces (considered up to isomeric bijections, denoted by $\mathcal{S}/_\sim$) can be endowed with a (pseudo-)metric itself, $d_{\mathrm{GH}} : \mathcal{S} \times \mathcal{S} \to (0, \infty)$, known as the Gromov-Hausdorff distance such that $d_{\mathrm{GH}}(X, Y) = 0$ if and only if X and Y are isometric. Informally, it quantifies how much one needs to 'distort' X to get Y. Furthermore, note that that space $(\mathcal{S}/_\sim, d_{\mathrm{GH}})$ is in fact a Polish space, see [LG13] for details.

Now, we can state our first convergence result.

Theorem 3.1 (Brownian map, [LG13], [Mie11])**.** *Uniform random planar maps (upon rescaling), viewed as random elements of $\mathcal{S}^a$, converge in law in the metric topology to an $\mathcal{S}$ (compact metric-space valued) random variable, also known as the Brownian map.*

[a] with metric the graph distance, for uniform triangulations and quadrangulations with a fixed number of vertices.

Remark.
- *The convergence of non-uniform random planar maps in the metric topology is open.*
- *For the peanosphere topology, however, convergence is known for both uniform and non-uniform random planar maps.*

One can take a look at `https://secure.math.ubc.ca/Links/Probability/images/gallery/brownian_map/brownian_map.html` for a nice simulation of a discrete approximation of the Brownian map based on sampling uniform triangulations on the sphere, and embedding it in $\mathbb{R}^3$. We will not focus on the Peanospere topology, and instead continue our discussion with the conformal topology.

3.2 conformal embedding of random planar maps

Before we continue our discussion of the conformal topology, we make a brief digression and talk about the topology of the rooted boundary of a planar map. For a planar map m, we define its boundary ∂, as the boundary of the root face of m . There are two possibilities, namely, the boundary ∂ can either be simple, i.e. going around the edges on the boundary of the root face, we never go back to the same edge, or non-simple (see Figure 4). In the former, case the boundary is homeomorphic to a circle (a very special case of the Jordan curve theorem from analysis).

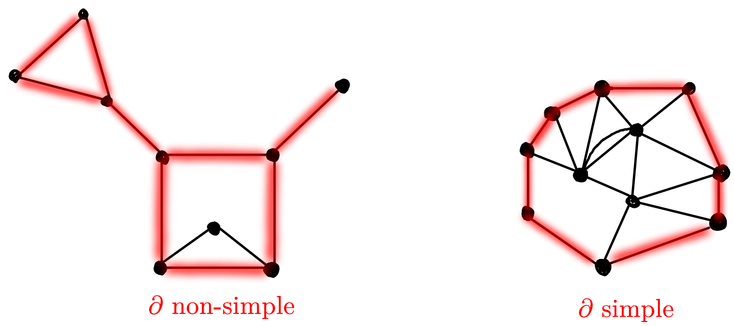


Figure 4: Illustration of two planar maps with topologically distinct boundaries.

Despite the different possible topologies of root boundaries of planar maps, asymptotically, this does not matter and the scaling limits will be the same. We will thus henceforth only consider random planar maps sampled with simple boundaries.

For such classes of planar maps, it is perhaps not surprising (as was also alluded to earlier) that one can define multiple notions of convergence that are underpinned by the notion of a conformal embedding, which we now turn to in more detail. But before discussing the particular of each way we will be 'conformally embedding' our random planar maps, we will outline the general framework which we will use to establish convergence.

Consider random planar maps $M_1, M_2, M_3, \cdots$ in the discrete topology, that is we suppose that $E(M_n) = n$ for $n \geq 1$. We then look for a discrete 'embedding' from the vertices of the random planar maps in to the closure of the unit disk, $\overline{\mathbb{D}}$,

$$\phi_n : V(M_n) \to \overline{\mathbb{D}}\,,$$

(see Figure 5).

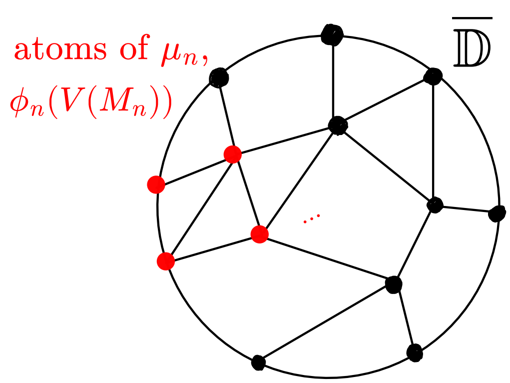


Figure 5: Illustration of discrete conformal embedding ϕ_n and the support of the measure it induces (red).

Now, let σ_n denote the rescaled vertex measures given by

$$\sigma_n(A) = \frac{1}{n}|A|\,, \qquad A \subseteq V(M_N)\,, \quad n \geq 1$$

and μ_n the pushforwards of the σ_n under the discrete embeddings ϕ_n. Then one easily sees that the μ_n have discrete support being exactly the images of the vertices of the planar maps in the unit circle (see Figure 5). Thus, we can view our random planar maps as random measures on the unit disk, which as a space can be endowed with a nice topology and enjoy certain compactness properties.

Particularly, let

$$\mathcal{S} := \{\mu : \mu \text{ is a finite Borel measure on } \overline{\mathbb{D}}\}$$

and endow it with the topology induced from the Lévy-Prokhorov metric $d_{\text{Prok}} : \mathcal{S} \times \mathcal{S} \to [0, \infty)$,

$$d_{\text{Prok}}(\mu, \tilde{\mu}) = \inf\{\epsilon > 0 : \forall A\, \mu(A) \leq \tilde{\mu}(A^\epsilon) + \epsilon\,, \tilde{\mu}(A) \leq \mu(A^\epsilon) + \epsilon\}\,,$$

where A^ϵ denotes the ϵ-neighbourhood of A. Moreover, the space $(\mathcal{S}, d_{\text{Prok}})$ is separable and complete, that is Polish.

We now say that the family of random planar maps M_n, $n \geq 1$ converges in the conformal topology (or under discrete conformal embedding) if there exists a random element of $\mathcal{S}$, μ, such that $\mu_n \implies \mu$, which is equivalent to convergence of $\int f\, \mathrm{d}\mu_n \implies \int f\, \mathrm{d}\mu$ as random variables for any bounded continuous $f : \overline{\mathbb{D}} \to [0, \infty)$.

We now answer the pressing question, namely, what is a discrete conformal embedding? To help us answer this question we start with answering the even more fundamental question, "*what is a conformal map*?", in no less than three different ways

Suppose $D, \tilde{D}$ are two Jordan domains and $\phi : D \to \tilde{D}$ is an orientation-preserving smooth bijection (see Figure 6).

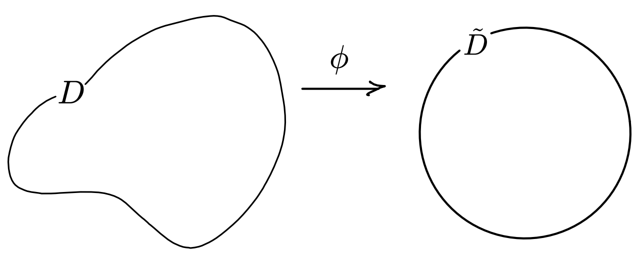


Figure 6: Illustration of conformal map between two domains D and $\tilde{D}$.

There are three equivalent definitions that determine whether the map ϕ is a conformal map. (a)

(a) The map ϕ is complex differentiable (with non-zero derivative).

(b) The map ϕ satisfies the Cauchy-Riemann equations.

(c) If W is a two dimensional Brownian motion, then $\phi \circ W$ is also a two-dimension Brownian motion (up to a time change).

(d) $\phi \circ \Gamma$ is a percolation scaling limit if Γ is a percolation scaling limit.

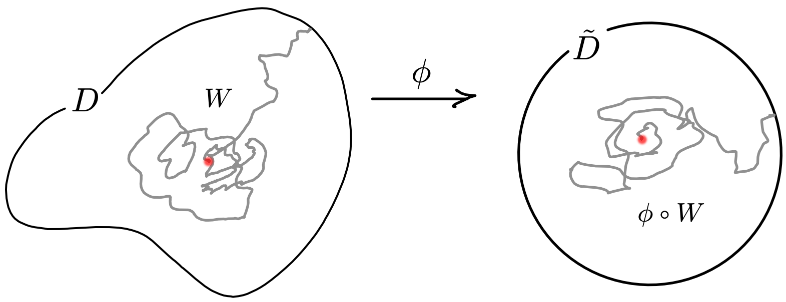


Figure 7: Illustration of Brownian motion definition for map ϕ to be conformal.

Having given multiple ways of characterising the notion of conformal embeddings, we now give procedures for how to discretise them. We first begin with discretising definition ⓒ, leading to the notion of Tutte embeddings.

A well-known fact from complex analysis is the existence of a unique conformal map $\phi : D \to \mathbb{D}$ such that $\phi(a) = 0$ and $\phi(b) = 1$, the condition on the points, 'fixed' the orientation of the map (see Figure 8).

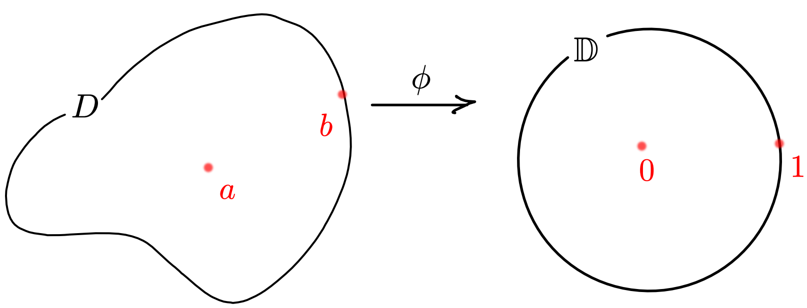


Figure 8: Illustration of of conformal map ϕ between two domains D, $\mathbb{D}$ mapping points a, b to $0, 1$ respectively in the unit disk $\mathbb{D}$.

One now proceeds to discretise the domain D by a lattice approximation of scale $1/n$ with interior D_n (Figure 9 partially indicated in gray) with boundary ∂D_n (Figure 9 partially indicated in red).

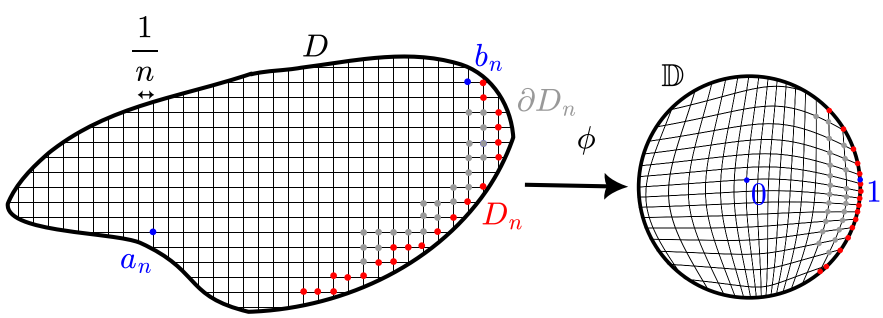


Figure 9: Illustration of Tutte embedding.

The goal is do define a 'discrete conformal' $\phi_n : D_n \cup \partial D_n \to \mathbb{D}$ such that in a reasonable sense, $\phi_n \approx \phi$ and $\phi_n(a_n) = 0$ and $\phi(b_n) = 1$ for distinguished points a_n, b_n, which are the lattice approximations to a, b respectively.

Now, we make this sense of 'discrete conformality' more precise. Let W^ν be a simple random walk on $D_n \cup \partial D_n$ with $W_0^\nu = \nu \in D_n \cup \partial D_n$, Z^x a planar Brownian motion starting from $x \in \mathbb{C}$,

$$\tau_n = \inf\{t \geq 0 : W_t^\nu \in \partial D_n\} ,$$

$$\tau_{\partial \mathbb{D}} = \inf\{t \geq 0 : Z_t^x \in \partial D\} ,$$

$$\tau = \inf\{t \geq 0 : Z_t^x \in \partial \mathbb{D}\} .$$

Donsker's Theorem (Theorem 2.3) gives the functional convergence

$$W^{a_n}|_{[0,\tau_n]} \overset{d}{\Longrightarrow} Z^0|_{[0,\tau_{\partial D}]} .$$

We want to construct, with ⓒ in mind, 'discrete conformal' maps $\phi_n, n \geq 1$ such that we have the functional convergence in law $\phi_n \circ W^{a_n}|_{[0,\tau_n]} \implies Z^0|_{[0,\tau]}$. However, the notion of convergence must allow for time-reparameterisations, in accordance with ⓒ, which leads to the following definition.

Consider the space of continuous finite paths in the plane

$$\mathcal{S} := \{\gamma : [0, T] \to \mathbb{C} : T > 0, \gamma \text{ continuous } \} ,$$

endowed with distance

$$d(\gamma_1, \gamma_2) := \inf_{\sigma} \sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(\sigma(t))| ,$$

where $\sigma : [0, T_1] \to [0, T_2]$ is an increasing bijection. Note that technically, d is a pseud-metric, but when considering the quotient of $\mathcal{S}$ modulo time-parameterisation, one obtains that indeed the quotient space $(\mathcal{S}/_\sim, d)$ is a Polish space. Informally, the distance thus defined is the uniform metric, 'modulo time-reparameterisation' [AB98, Lemma 2.1].

Note, the distance d is <u>not</u> equivalent to the Hausdorff distance on traces of $\gamma \in \mathcal{S}$ since one can consider paths with different parameterisations (see Figure 10).

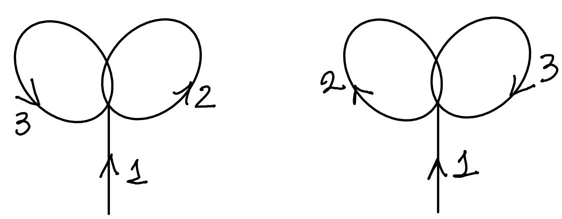


Figure 10: Illustration of paths $\gamma_1, \gamma_2 \in \mathcal{S}$ with $d_{\text{Hausdorff}}(\gamma_1, \gamma_2) = 0$, yet $d(\gamma_1, \gamma_2) > 0$.

Now, <u>define</u> $\phi_n|_{\partial D_n}$ such that $\phi_n \circ W^{a_n}(\tau_n)$ has the desired law, that is

$$\mathbb{P}(W^{a_n}(\tau_n) \in \text{arc}(D_n \cup \partial D_n; b_n, \nu)) := \mathbb{P}(Z^0(\tau) \in \text{arc}(\mathbb{D}; 1, \phi_n(\nu))) ,$$

which turns out to uniquely characterise $\phi_n(\nu)$, since the probabilities above are strictly monotone with respect to the length of the 'arc'. To extend ϕ_n to the interior of the lattice approximation, D_n, one simply requires that $\phi_n \circ W^{a_n}$ is a martingale, or equivalently,

$$\phi_n(\nu) = \frac{1}{(\deg(\nu) = 4)} \sum_{\omega \sim \nu} \phi_n(\omega) ,$$

that is if and only if ϕ_n is discretely harmonic, which can always be done, by a classical result from linear algebra, or the observation that the optional stopping theorem gives

$$\phi_n(\omega) = \mathbb{E}[\phi_n(W^\nu(\tau_n))]\,,$$

which yields the desired uniqueness, given a prescription of boundary values $\phi|_{\partial D_n}$.

More generally, the Tutte embedding constructed above is well defined for more general planar maps with root boundary having the disk topology. Given the graph structure of a planar map with a boundary homeomorphic to the boundary of a disk, one can construct using the same 'discrete harmonic' construction as above a 'conformal embedding' of the graph in the unit disk (with the root face boundary contained in a 'monotonic' way on the boundary of the disk).

Now, for a family of random planar maps M_n in the discrete topology with root boundaries having disk topologies, discrete embeddings $\tilde{\phi}_n$ from planar maps into the unit disk, with W^ν denoting a random walk in $V(M_n)$ started from a vertex ν in $V(M_n)$, set

$$\lambda_n^\nu = \text{law of } \tilde{\phi}_n \circ W^\nu|_{[0,\tau_n]}$$

$$\lambda_{W_n}^\nu = \text{law of } \tilde{\phi}_n \circ W^\nu|_{[0,\tau_n]} \text{ given } W_n$$
$$\lambda^x = \text{law of } Z^x|_{[0,\tau]}$$

where τ_n, τ denote the hitting times of ∂D_n and $\partial\mathbb{D}$ by the random walks W on $V(M_n)$ and Brownian motion Z^x (started at x in the interior of $\mathbb{D}$) respectively.

We now say $\lambda_n^{a_n}$ converges to λ^0 in the <u>annealed</u> sense if

$$d_{\text{Prok}}(\lambda_n^{a_n}, \lambda^0) \overset{n\to\infty}{\longrightarrow} 0$$

and in the <u>quenched</u> sense (which is stronger than annealed convergence) if

$$d_{\text{Prok}}(\lambda_{W_n}^{a_n}, \lambda^0) \overset{n\to\infty}{\Longrightarrow} 0\,.$$

We are now in a position to prove a proposition that gives conditions under which embeddings of random planar maps $(M_n)_{n\in\mathbb{N}}$ (with disk topology) $\tilde{\phi}_n$ into the unit disk are close to the Tutte embeddings associated to the planar maps.

Proposition 3.2. *Let $(M_n)_{n\in\mathbb{N}}$ be planar maps with (with rooted boundary having disk topology) and let $\tilde{\phi}_n : V(M_n) \to \mathbb{D}$ be discrete embeddings into the unit disk. Suppose the following hold.*

(I)

$$\sup_{\nu\in V(M_n)} d_{Prok}(\lambda_{W_n}^\nu, \lambda^{\tilde{\phi}_n(\nu)}) \implies 0\,;$$

(II)

$$\tilde{\phi}_n(\nu) \in \partial\mathbb{D} \qquad \textit{for all } \nu \in \partial V(M_n)\,;$$

(III) *If $\nu_1, \nu_2, \cdots$ are ordered counter-closckwise on ∂M_n then $\tilde{\phi}_n(\nu_1)$, $\tilde{\phi}_n(\nu_2), \cdots$ are ordered counter-clockwise on $\partial\mathbb{D}$.*

Then, $\|\tilde{\phi}_n - \phi_n\|_\infty \to 0$ as $n \to \infty$, where ϕ_n are appropriate Tutte embeddings.

Remark. *Conditions (II), (III) are necessary (can produce counterexamples with rescaling, or by considering $\phi_n(\nu) \mapsto (\phi_n)^2(\nu)$, which is still conformal, shows that (III) is necessary).*

(Sketch). Fix $\nu \in V(M_n)$. Then there are two cases to consider.

(i) $\nu \in \partial V(M_n)$. Then, we can estimate using the fact that the images of the random walks under a_n 'look like' standard Brownian motion to obtain a uniform comparison of the harmonic measures on the boundary of 'arcs' of images of discrete 'arcs' of the boundary of $V(M_n)$ using the 'monotonicity' of both maps on the boundary.

$$\begin{aligned} &\mathbb{P}(Z^0(\tau) \in \operatorname{arc}_{\mathbb{D}}(1, \tilde{\phi}_n(\nu))) \\ \overset{\text{(I)}}{\approx}\; &\mathbb{P}(\tilde{\phi}(W^{a_n}(\tau_n)) \in \operatorname{arc}_{\mathbb{D}}(1, \tilde{\phi}_n(\nu))|W_n) \\ \overset{\text{(II) \& (III)}}{=}\; &\mathbb{P}(W^{a_n}(\tau_n) \in \operatorname{arc}_{M_n}(b_n, \nu)|W_n) \\ \overset{\text{Tutte definition}}{=}\; &\mathbb{P}(Z^0(\tau) \in \operatorname{arc}_{\mathbb{D}}(1, \phi_n(\nu)))\,. \end{aligned}$$

This now allows one to compare $\tilde{\phi}_n$ and ϕ_n uniformly on the boundary $\partial V(M_n)$.

(ii) : $\nu \in V(M_n) \setminus \partial V(M_n)$. Then we have

$$\begin{aligned} \mathbb{E}[Z^{\tilde{\phi}_n(\nu)}(\tau)] \;&\overset{\text{OST}}{=} \tilde{\phi}_n(\nu) \\ &\overset{\text{(I)}}{\approx} \mathbb{E}[\tilde{\phi}_n(W^{\nu}(\tau_n))] \\ &\overset{\text{case }(i)}{\approx} \mathbb{E}[\phi_n(W^{\nu}(\tau_n))] \\ &\overset{\text{Tutte definition}}{=} \phi_n(\nu)\,. \end{aligned}$$

□

Remark. *Similar proposition holds for percolation and the Cardy-Smirnov embedding.*

How does one discretise definition (a)? The approach we will take is inspired by the seminal work of Smirnov, [Smi01], that involves constructing a conformal map between domains using percolation on the triangular lattice. This sort of argument is then translated into the setting of embedding random planar maps in the simplex using the so-called, Cardy-Smirnov embedding.

Now, consider a domain $D \subseteq \mathbb{C}$ in the plane with three distinguished points a, b, c on its boundary ∂D, labelled counter-clockwise. Consider the discretisation of D, $D_n \cup \partial D_n$, using a triangular lattice of scale $1/n$ with distinguished points a_n, b_n, c_n on ∂D_n again, labelled counter-clockwise, being the lattice approximations to a, b, c (see Figure 11. Denote by (a_n, b_n) the set of boundary points on ∂D_n between a_n, b_n and define (b_n, c_n) and (c_n, a_n) similarly. Now, let $\omega : D_n \cup \partial D_n \to \{\text{red}, \text{green}\}$ be a colouring of the lattice approximation $D_n \cup \partial D_n$ such that boundary values are coloured red and the interior ones are coloured uniformly and independently of each other either red or green.

Now, for a vertex $\nu \in D_n \cup \partial D_n$, consider the events $E_{a_n}(\nu)$ that there exists a simple path P on $D_n \cup \partial D_n$ such that

(a) P has one endpoint in (c_n, a_n) and one endpoint in (a_n, b_n), while all other vertices of P are inner red vertices, and

(b) either $v \in P$ or v is on the same side of P as the edge a_n.

See Figure 12. We define the events $E_{b_n}(\nu)$ and $E_{c_n}(\nu)$ similarly.

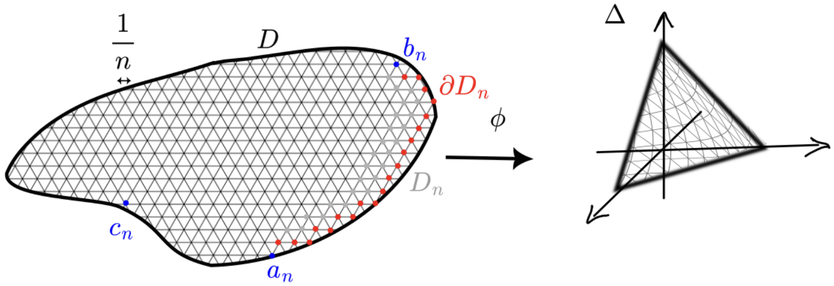


Figure 11: Illustration of lattice decomposition $D_n \cup \partial D_n$ of domain D in the plane with three distinguished vertices a_n, b_n, c_n and its image under the conformal map ϕ in the unit simplex $\Delta := \{(x, y, z) : x + y + z = 1,\ x, y, z > 0\}$.

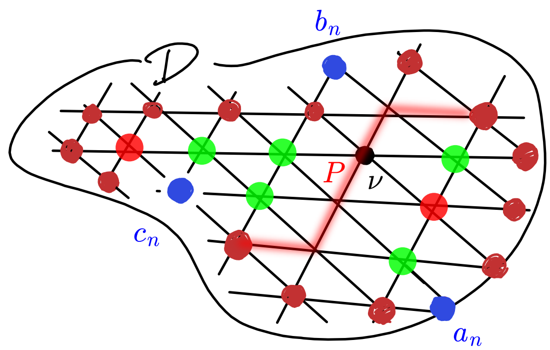


Figure 12: Illustration of the event the $E_{a_n}(\nu)$ appearing in the definition of the Cardy-Smirnov embedding.

Consider the equilateral triangle $\Delta := \{(x, y, z) : x + y + z = 1,\ x, y, z > 0\}$ and let $\overline{\Delta}$ denote its closure. The Cardy-Smirnov embedding of (M, a, b, c) is the function $p_n : D_n \cup \partial D_n \to \overline{\Delta}$ given by

$$p_n(\nu) = \frac{(\mathbb{P}[E_{a_n}], \mathbb{P}[E_{b_n}(\nu)], \mathbb{P}[E_{c_n}(\nu)])}{\mathbb{P}[E_{a_n}(\nu)] + \mathbb{P}[E_{b_n}(\nu)] + \mathbb{P}[E_{c_n}(\nu)]},$$

where $\mathbb{P}$ is the law of the percolation ω given the lattice approximation of D (suppressing dependence on $n \geq 1$).

In [Smi01], Smirnov showed that (supposing D is simply-connected and a Jordan domain), then the mappings p_n (appropriately extended to D approximate the Riemann mapping from D to Δ mapping a, b, c on D to appropriate extreme points on the simplex $\overline{\Delta}$). TheIn particular, the limits of the p_n as $n \to \infty$ are conformal invariants of the domain and marked points $a, b, c \in \partial D$.

Now, as in the setting of the Tutte embeddings, one can verbatim extend the setup to any planar map M that is a triangulation with simple boundary and having three distinguished vertices on the root boundary.

4 convergence of random planar maps

Now, having established our modes of convergence of random planar maps, all 'uniform' in some sense, we turn to the question of the existence of scaling limits. A main research direction is that of existence and universality of scaling limits of random planar maps, constituting the source of many conjectures and open problems in the field. First, we give a brief definition of some of the continuum objects that appear in some of the already-established scaling limits, starting with the Gaussian free field.

Fix a domain $D \subseteq \mathbb{C}$. Suppose one wants to define a Gaussian field '$h : D \to \mathbb{R}$' specified by setting $\mathbb{E}h(x) = 0$ and $\mathrm{Cov}(h(x), h(y)) = G(x, y)$ for $x, y \in D$, where $G(\cdot, \cdot)$ is the green's function of $-\frac{1}{2\pi}\Delta$, which is $\int_{(0,\infty)} p_t^D(x, y)\,\mathrm{d}t$, where

$$p_t^D(x, \cdot) = \text{ density of } Z^x(2(t \wedge \tau)))\,,$$

where Z^x is a planar Brownian motion starting from $x \in D$ and τ is the first hitting time of the boundary of ∂D[1] . Then, one has to leading order

$$G(x, y) = \log|x - y|^{-1} + \log(CR(x, D)) + o(1)\,, \qquad \text{as } y \to x\,,$$

where $C > 0$ is some constant and $R(x, D)$, is known as the 'conformal radius' of $x \in D$. This gives that G is singular along the diagonal (and so we would have that the variance of the field at any point would be infinity) which means one cannot construct a Gaussian field with the above covariance structure in the usual way and so h cannot be realised as a bonna-fide (random) function. Despite h being too irregular to be a function, one can make sense of h as a random linear functional on signed measures on D ('local averages of h make sense and are well-defined'). A sign as to why this might be possible is that by Fubini

$$\int_D G(x, y)\,\mathrm{d}y = \int_{(0,\infty)} \int_D p_t^D(x, y)\,\mathrm{d}y\,\mathrm{d}t = \int_{(0,\infty)} \mathbb{P}(Z^x(t) \in D, \tau \geq t)\,\mathrm{d}t \leq \int_{(0,\infty)} \mathbb{P}(\tau \geq t)\,\mathrm{d}t = \mathbb{E}\tau_{\partial D} < \infty\,.$$

Now, before we make the formal definition of the Gaussian Free Field, we make some preliminary definitions. Set

$$\mathcal{M}_D := \{\mu : \mu \text{ finite Borel measure supported in } D \text{ such that } \int_{D\times D} G(x, y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty\}\,,$$

$$\mathcal{M} := \{\rho : \rho = \rho_+ - \rho_-\,, \rho_\pm \in \mathcal{M}_D\}\,.$$

Then we are ready to define the Gaussian free field.

Definition 4.1 (Continuum Gaussian free field (GFF))**.** *Let $D \subseteq \mathbb{C}$ be a bounded domain, when we say the process $(h_\rho)_{\rho\in\mathcal{M}}$ is a Gaussian free field in D if it is a centred Gaussian process with covariance function*

$$\Sigma(\mu, \nu) = \int_{D\times D} G_D(x, y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,, \qquad \mu, \nu \in \mathcal{M}\,.$$

Note one can check that the function $\Sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is indeed a covariance function, that is for any $n \geq 1$ and real $\lambda_1, \cdots \lambda_n$,

$$\sum_{1\leq i,j\leq n} \Sigma(\mu_i, \mu_j) \geq 0\,.$$

Thus, Kolmogorov's extension theorem, we can construct the Gaussian free field as a stochastic process. Note (by checking that second moments of differences vanish) for any $\lambda \in \mathbb{R}$, $\rho, \mu, \nu \in \mathcal{M}$,

$$h_{\lambda\rho} = \lambda h_\rho\,, \qquad h_{\mu+\nu} = \qquad h_\mu + h_\nu$$

almost surely. Note this gives the alternative characterisation of the Gaussian free field in terms of the variance condition and linearity.

However, note this does not allow us to say much about the GFF simultaneously for all $\rho \in \mathcal{M}$ (and so we cannot speak of it as a random continuous function on $\mathcal{M}$ in any meaningful sense). This picture changes once one restricts the GFF to 'nicer' subsets of $\mathcal{M}$, whence one obtains that the GFF can be realised as a random generalised function, see below.

[1] One is forced to consider this covariance structure when considering a lattice approximation to the domain D and one considers scaling limits of the discrete Gaussian Free field, see [WP21].

The Gaussian free field can be viewed in some sense as the canonical generalisation of Brownian motion in two dimensions. This is partly because it enjoys the **domain Markov property**.

Proposition 4.2 (Domain Markov property, [WP21]). *Let $D \subseteq \mathbb{C}$ be a domain in the plane and A some compact subset of $\overline{D}$ such that $O := D \setminus A$ also has regular boundary, and h be a GFF in D. Then, there are two independent Gaussian processes h_A, h^A such that*

- *h^A has the law of the Gaussian free field in O;*
- *there exists a version of h_A such that its restriction $h_A|_O$ is almost surely equal to a harmonic function in O.*

This means intuitively that the GFF conditionally on its values outside some domain is given by its 'boundary values' on that domain (since harmonic functions are determined by their boundary values) and an independent components which is distributed as a GFF in the domain. One palpably sees the connection with the usual Markov property enjoyed by one dimensional Brownian motion.

We now briefly (re)-construct the GFF on a bounded domain $D \subseteq \mathbb{C}$, as a random generalised function by its representation as a 'random Fourier series'. By standard rezults from functional analysis, see [Eva22, Chapter 6], the Hilbert space of square-integrable functions $L^2(D)$ has an orthonormal basis of functions $(\phi_j)_{j\in\mathbb{N}}$ that are eigenfunctions of $-\Delta$. Moreover, with $(\lambda_j)_{j\geq 1}$ denoting the corresponding eigenvalues and $N(\lambda)$ counting the number of eigenvalues less than or equal to λ^2 , one has from Weyl's law,

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = c \cdot \mathrm{vol}(D)\,,$$

for some positive constant $d > 0$. An easy computation gives with Σ the covariance matrix as in 4.1,

$$\Sigma(\phi_i, \phi_j) = \frac{1}{\lambda_i}\mathbf{1}_{i=j}\,.$$

Thus, the family of random variables $(\mathcal{N}_j := \sqrt{\lambda_j}\Gamma(\phi_j))_{j\in\mathbb{N}}$ is a family of independent standard Gaussian random variables. Now, for $\mu \in \mathcal{M}$, one immediately obtains by properties of the Green's function $G(\cdot,\cdot)$

$$\sum_{j\in\mathbb{N}} \lambda_j^{-1} \int_D \phi_j(x)|\mu|(\mathrm{d}x) < \infty\,.$$

Thus, setting

$$\tilde{\Gamma}(\mu) := \sum_{j\in\mathbb{N}} \mathcal{N}_j \lambda_j^{-1\frac{1}{2}} \int_D \phi_j(x)\mu(\mathrm{d}x)\,, \qquad \mu \in \mathcal{M}\,,$$

we see from the above that $\tilde{\Gamma}$ has the law of a GFF on D. Again, one cannot conclude much about $\tilde{\Gamma}(\cdot)$, as a random linear function (simultaneously for all entries) on $\mathcal{M}$. However, for $s > 0$ (determined by Weyl's law) restricting to the class of functions in $L^2(D)$,

$$\mathcal{H}^s := \left\{ f \in L^2(D) : \|f\|^2_{\mathcal{H}^s} := \sum_{j\in\mathbb{N}} \lambda_j^s \left(\int_D f(x)\phi_j(x)\,\mathrm{d}x \right)^2 < \infty \right\}\,,$$

(which is a Hilbert space endowed with the above norm), one can define $\tilde{\Gamma}(f)$ simultaneously for all $f \in \mathcal{H}^s$, since one can control the sums uniformly and actually prove the estimates

$$|\tilde{\Gamma}(f) - \tilde{\Gamma}(g)| \leq C \cdot \|f - g\|_{\mathcal{H}^s}\ , \text{ for } f, g \in \mathcal{H}^s\,,$$

for some positive $C > 0$. This establishes that for any $s > 0$, the GFF can be realised as a random element of the continuous dual of the Hilbert space $\mathcal{H}^s$. Since by the Riesz-representation theorem if this were true for any $\mathcal{H}^s$, $s \leq 0$, one could realise the GFF as a bonna-fide random element of $L^2(D)$, we see that the

[2]which is well-defined since the inverse operator of $-\Delta$, which itself exists, is compact and self-adjoint and the rest is a direct application of the spectral theorem for copact, self adjoint operator. See the Appendix in [Eva22] for details.

GFF is 'barely' a function in two dimensions. In one dimension, however, the same arguments actually give a random function with a continuous modification, which is readily identified with Brownian motion, which is another motivation for considering the GFF as a 'Brownian sheet'.

Now, fix a GFF Γ on some domain $D \subseteq \mathbb{C}$, we will be considering 'spherical averages' of the GFF on balls in the domain and show that one can construct a Brownian motion from time-reparameterisation of said time averages.

More precisely, suppose that z_0 is fixed in D and let r_0 be a positive real smaller than $\mathrm{dist}(z_0, \partial D)$. Let $\lambda_{z_0,r}$ denote the uniform (rescaled Lebesgue measure) probability measure on the boundary of the ball $B(z_0, r)$. Noting that $\lambda_{z_0,r} \in \mathcal{M}$, we define for all $r \leq r_0$

$$\gamma(z_0, r) = \Gamma(\lambda_{z_0,r})\,.$$

Observe by harmonicity properties of the Green kernel, by computing covariances, one can compute that the process $(r \mapsto (\gamma(Z_0, r) - \gamma(z_0, r_0)))$ is independent of any $\gamma(\mu)$ with $\mu \in \mathcal{M}$, supported outside of $\overline{B(z_0, r)}$. Now, one can compute using the Markov property 4.2, for any $0 < r \leq r; \leq r_0$,

$$\mathbb{E}[(\gamma(z_0, r) - \gamma(z_0, r'))^2] = |r - r'|\,.$$

Thus, the re-parameterised process $B \coloneqq (\gamma(Z_0, r_0\mathrm{e}^{-u}) - \gamma(z_0, r_0\mathrm{e}^{-u}))_{u \geq 0}$ is a Gaussian process with the same finite dimensional distributions as one-dimensional Brownian motion (for a cartoon illustration see Figure 13). Now one can obtain a continuous modification by an application of Kolmogorov's continuity lemma. Furthermore, B is independent of the sigma algebra

$$\{\Gamma(\mu) : \mathrm{supp}(\mu) \cap \overline{B(z_0, r)} = \emptyset\}\,.$$

For a more detailed discussion of the continuum as well as discrete GFFs, see the book [WP21].

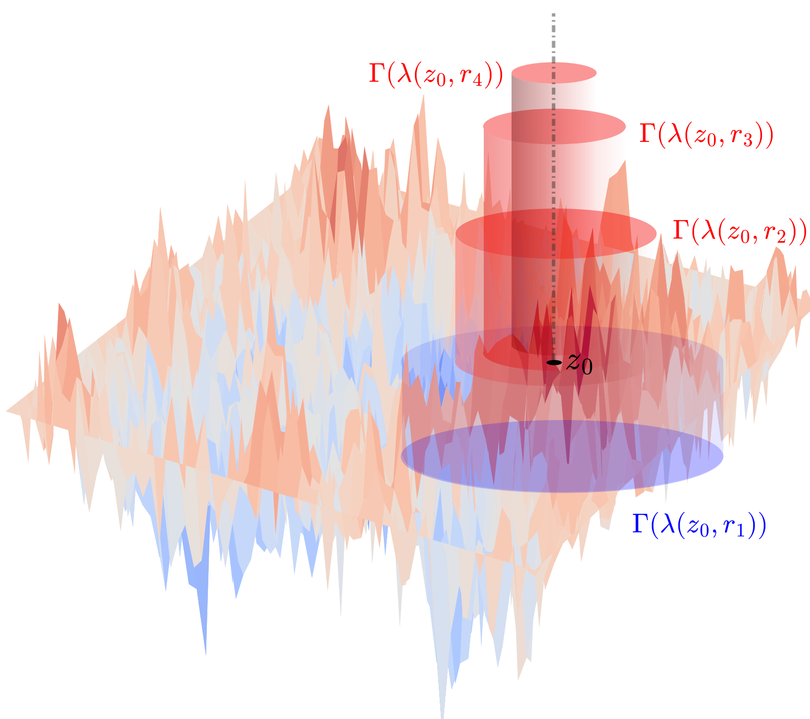


Figure 13: Cartoon illustration of 'disk averaging' for the GFF using a discrete approximation thereof on D a bounded square with concentric disks with radii $r_1 > r_2 > r_3 > r_4 > 0$ centred at $z_0 \in D$. Red colours indicate positive values, and blue colours indicate positive values.

Random surfaces were also studied in a non-rigorous way in the physics literature (see [Pol81]) in attempts to define a theory of integration over all possible surfaces, in a string-theoretic generalisation to the Feynman integral. This naturally led to the question of studying measures on all such surfaces, and thus probability measures on surfaces. In coordinates, one could try to formally consider random metrics

$$\text{“ } \mathrm{d}s^2 = \mathrm{e}^{\gamma h}(\mathrm{d}x^2 + \mathrm{d}y^2)\text{”}$$

on some domain $D \subseteq \mathbb{C}$, where γ is some parameter and h the Gaussian free field on D (for more motivation on why this functional form is considered, see the review article [Gwy21]). However, this is ill-defined as the GFF h cannot even be realised as a function. One aspect one can extract and try to make rigorous is that of the 'area' measure associated to the surface. This leads to the construction of the Liouville quantum gravity (LQG) surface with area measure $\mathrm{e}^{\gamma h}\,\mathrm{d}A$, with γ, h as before, where $\mathrm{d}A$ is the Lebesgue measure on D. It turns out that LQG area measures can be constructed by first regularising the Gaussian free field (by convolution with some smooth mollifier giving random smooth functions $h_\varepsilon, \varepsilon > 0$) and considering the limit in law of random measures

$$\mu_\gamma := \lim_{\epsilon \to 0} \varepsilon^{\gamma^2/2} \mathrm{e}^{\gamma h_\varepsilon}\,\mathrm{d}A\,.$$

The parameter $\gamma \in (0, 2)$ controls the 'roughness' of the LQG surface, with $\gamma = 0$ corresponding to flat space and larger values of γ lead to more 'fractal-like' behaviour departing from the classical Euclidean picture.

Moreover, there are deterministic relationships (known as KPZ formulas in the literature)[3] between the Hausdorff (Euclidean) dimensions of fixed Borel subsets $A \subseteq D$ and their 'quantum' counterparts which correspond to Hausdorff dimensions of sets with the underlying metric being a random one, associated in a canonical sense to γ-LQG surfaces. For more details, see [GP19] and for a more in-depth introduction, see the article [Gwy21].

We now state the first claim to universality in this field, namely the convergence of a large class of random planar maps to the LQG surfaces.

Conjecture 4.3. *A number of random planar maps converge to γ-Liouville quantum gravity (γ-LQG)[a] under discrete conformal embeddings[b].*

[a] recall the parameter $\gamma \in (0, 2)$ has to do with the fact that the LQG area measure is a model for random 'Riemannian surfaces parameterised by a metric tensor with γ-dependent exponent $\mathrm{e}^{\gamma h}$, where h is a Gaussian Free Field (GFF)'

[b] Which there are lots of ways to define.

Having given a brief account of the continuum objects of interest, we are now in a position to state various convergence results, making the case for conjecture 4.3.

Theorem 4.4 ([GMS21b])**.** *Conjecture 4.3 is true for mated CRT map under the Tutte embedding and $\gamma \in (0, 2)$.*

Theorem 4.5 (N. Holden '25)**.** *Conjecture 4.3 is true for tree-weighted maps under the Tutte embedding, and $\gamma = \sqrt{2}$.*

Theorem 4.6 ([HS21])**.** *Conjecture 4.3 is true for uniform triangulations under the Cardy-Smirnov embedding and $\gamma = \sqrt{8/3}$.*

We briefly delineate the proof strategies for the above theorems, starting with Theorems 4.4 and 4.5.

Consider a family of random planar maps $(M_n)_{n\in\mathbb{N}}$ in the discrete topology with simple boundaries. Then one can construct some (other than the Tutte) embeddings $\tilde{\phi}_n : V(M_n) \to \mathbb{D}$ such that

[3] Not to be confused with the KPZ universality class.

(1) the pushforward measures of the rescaled counting measures on $V(M_n)$, $\tilde{\mu}_n$ converge in law as random measures to the the Liouville quantum gravity area measure μ;

(2) the uniform distance of the embeddings $\tilde{\phi}_n$ and the respective Tutte embeddings ϕ_n converges to zero in distribution (which implies $d_{\text{Prok}}(\mu_n, \tilde{\mu}_n) \implies 0$, where μ_n is defined as $\tilde{\mu}_n$, but with respect to the Tutte embeddings ϕ_n).

How do we pick the embeddings $\tilde{\phi}_n$ for Theorem 4.5? The answer is to use the Mullin bijection (2.1) and the continuum mating of trees procedure that couples LQG measures and planar Brownian motion (by considering lengths boundaries of the image of space-filling SLE curves in the disk reparameterised by LQG area). For more details of this construction, the interested reader can take a look at [DMS20], [HS25, Figure 3.3].

For simulations and graphic depictions of Tutte embeddings of mated CRT maps (in [Gwy21] can be found in `http://statslab.cam.ac.uk/~jpm205/tutte-embeddings.html`).

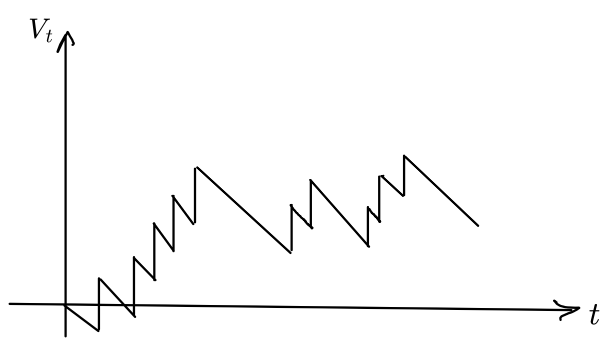
$$
\begin{array}{ccc}
\text{Walk} & \xrightarrow{n\to\infty} & \text{planar BM} \\
\text{Mullin}\big\downarrow & & \big\downarrow \text{mating of trees embedding } p(t)\in\overline{\mathbb{D}} \\
(M,T) & & \sqrt{2}\text{-LQG}
\end{array}
$$

Figure 14: The diagram above illustrates the correspondence between the tree-weighted planar maps to LQG through the Mullin bijection, which is coupled to a planar Brownian motion through a KMT-type coupling (see below), which itself is taken to be coupled to the LQG using the continuum mating of trees procedure. This gives rise to discrete embeddings $\tilde{\phi}_n$ by taking vertices through the Mullin bijection to a corresponding walk time index which is then mapped into the disk through the mating of trees embedding p.

Now, to prove the convergence of the discrete embeddings $\tilde{\phi}_n$ to the $\sqrt{2}$−LQG area measure, the final missing ingredient is the arrow between 'Walk' and planar Brownian motion. For this one needs a 'strong' coupling between the walk indices produced from the Mullin bijection applied to the random planar maps (M, T) and 2D Brownian motion. This is achieved by means of KMT-type (Komlós-Major-Tusnády, [KMT75]), which we briefly describe below.

For $a > 0$, one can obtain a coupling of

- $(V_t)_{t\geq 0}$, $V_t = P_t - at$, where $P =$ a rate a Poisson process;



- $(Y_t)_{t\geq 0}$ a variance a one dimensional Brownian motion;

such that one has the following.

(a) For all $\alpha > 1$, there exists $C > 1$ such that

$$\mathbb{P}\left(\sup_{z\in[0,2^k]} |V_t - Y_t| > Ck\right) \leq 2^{-\alpha k}\,;$$

(b) there exists ϵ such that for $k \geq 1$, there exist functions F_k such that

$$\mathbb{P}\left(V_{[0,2^k]} \neq F_k(Y_|[0, 2^{k+m}])\right) \leq 2^{-\varepsilon m}\,.$$

Informally, the above mean that the Brownian motion Y is 'determining V in a local sense with a very high probability'.

This coupling then allows us to construct couplings of

- $(\tilde{W}_t)_{t\geq 0}$, $\tilde{W}_t = W_{P_t}$, where P is a rate two Poisson process, W is a random walk with iid. and uniform steps $\{(\pm 1, 0), (0, \pm 1)\} = \{v_1, v_2, v_3, v_4\}$;
- $(Z_t)_{t\geq 0}$ a planar Brownian motion.

This is done by defining

$$\tilde{W}_t = \sum_{i=1}^{4} P_t^j v_j\,, \qquad t \geq 0\,,$$

where P^j are independent rate 1/2 Poisson processes, which allows one express

$$\tilde{W}_t = \sum_{i=1}^{4} \underbrace{(P_t^j - 1/2t)}_{\text{couple with 1D Brownian motion with variance 1/2}} v_j\,, \qquad t \geq 0\,.$$

Now, the missing link is to extablish the convergence of the Tutte embeddings ϕ_n and the discrete embeddings $\tilde{\phi}_n$. Recall Proposition 3.2 gives sufficient conditions to prove (2), in particular condition (I). One is naturally led to consider random walks in random environments (given by the graph structure of the random planar maps). A natural next question is how one might achieve the quenched convergence in (I) of Proposition 3.2 in the setting of random planar maps.

Note in the context of random planar maps, one does not expect the random environments (embedded in the plane) will have a typical length scale or enjoy invariant (in law) under translations, where convergence results are more classical. One typically deals with 'fractal' environments. Yet, it turns out they enjoy some structure, namely, that of 'translation invariance modulo scaling', that fits into the following more general framework.

For a random graph with vertices and edges (V, E) respectively, where $V \subseteq \mathbb{C}$ is locally finite, such that (informally see [GMS21a, Definition 1.2])

- V is translation invariant modulo scaling, i.e. 'allowed to scale by random constants before and after scaling'
- ergodic (modulo scaling)
- well-connected
- high-degree vertices and long edges unlikely,

then the following holds.

Theorem 4.7 (Theorem 1.5 [GMS21a]). *Let $\mathcal{M}$ be an 'ergodic scale-free' (informally in the sense above), random environment. Then the simple random walk W on $\mathcal{M}$, converges in the quenched sense to a planar Brownian motion modulo time-parameterisation. That is, the conditional law of the walk given the environment converges in distribution to planar Brownian motion (module time-parameterisation).*

Now, how do the random environments we obtain using the planar embeddings $\tilde{\phi}_n$ compare to the above?

Theorem 4.8 ([GMS21b]). *Pick $z_0 \sim \text{Leb}(\mathbb{D})$ and rescale by 'zooming in' near z_0 while $n \to \infty$. Then, a local limit of the planar embeddings exists and satisfies the assumptions of Theorem 4.7.*

Again, anlogously to the above, Theorem 4.6 can be viewed as a quenched convergence result for percolation on random planar maps, see [HS21] for more details.

On this note, we end out sketch of the proof of the main convergence results of interest and hope to have imbued the reader with some sense of the workings and richness of random planar maps and encourage them to further pursue their investigations in this wonderful subject.

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